Problem 7.7.2 (a)

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$

with $\frac{\partial u}{\partial r}(a, \theta, t) = 0$

and initial cond. $u(r, \theta, 0) = 0$ $\quad \frac{\partial u}{\partial t}(r, \theta, 0) = \beta(r) \cos 5\theta$

separate variables: $u(r, \theta, t) = f(r)\, g(\theta)\, h(t)$

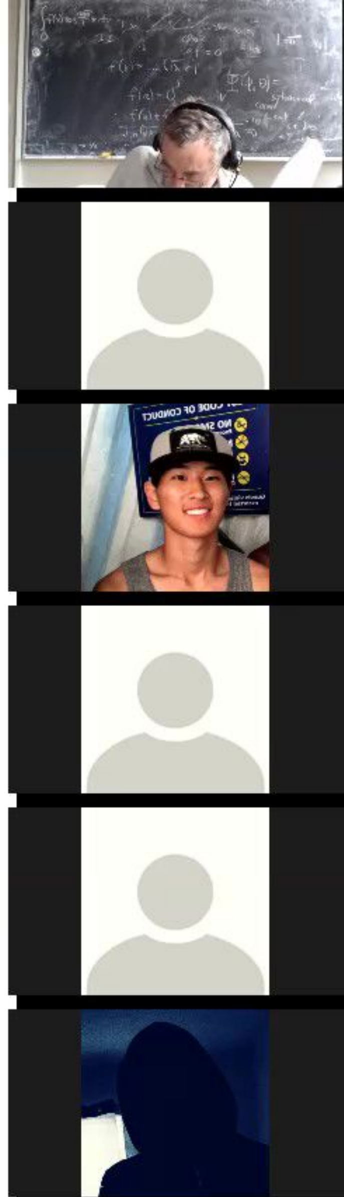
got 3 ODE's

$$r \frac{d}{dr}\left(r \frac{df}{dr}\right) + (\lambda r^2 - m^2) f = 0$$

$$h''(t) = -\lambda c^2 h(t)$$

$$g''(\theta) = -m^2 g(\theta)$$

new: $\frac{\partial u}{\partial r}(a, \theta, t) = 0 \implies \boxed{f'(a) = 0}$



showed in class:

$$f(r) = J_m(\sqrt{\lambda}\, r) \quad \text{solves ODE for } f$$

J_m Bessel function of first kind.

need to find values for λ

$$\Rightarrow \quad f'(a) = 0$$

determines eigenvalues

$\sqrt{\lambda_{mn}}\, a = z_{mn}$

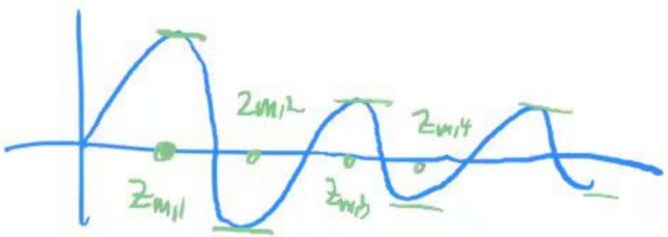
$\Rightarrow \boxed{\sqrt{\lambda_{mn}} = z_{mn}/a}$

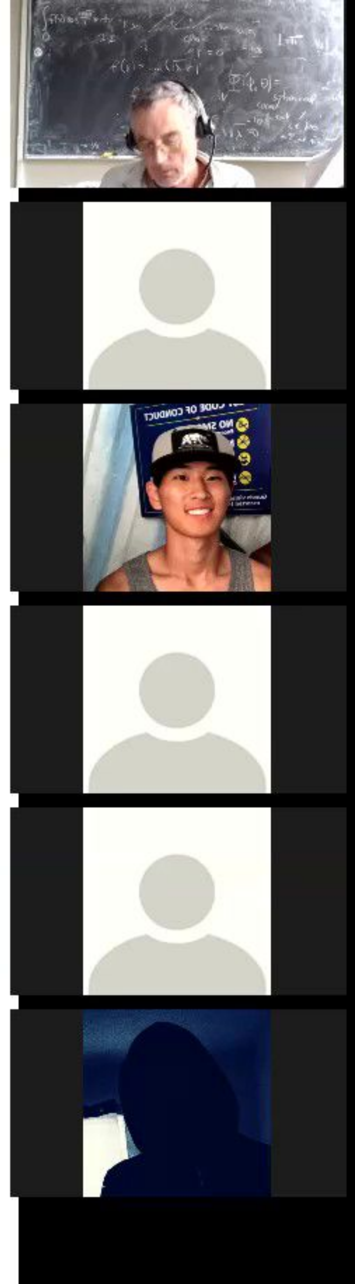
$$\frac{d}{dr} J_m(\sqrt{\lambda}\, r) = \sqrt{\lambda}\, J_m'(\sqrt{\lambda}\, r)$$

recall: J_m is oscillating $\Rightarrow$ have infinitely many zeros for J_m'

label them

$(z_{m,1}, z_{m,2}, z_{m,3}, \dots)$





$\Rightarrow$ general solution given by

$$\sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m\left(\sqrt{\lambda_{m,n}}\, r\right)\left(a_m \cos m\theta + b_m \sin m\theta\right) \underbrace{\left(a_{mn} \cos \sqrt{\lambda_{mn}}\, ct + b_{mn} \sin \sqrt{\lambda_{mn}}\, ct\right)}_{= a_{mn} \text{ for } t=0}$$

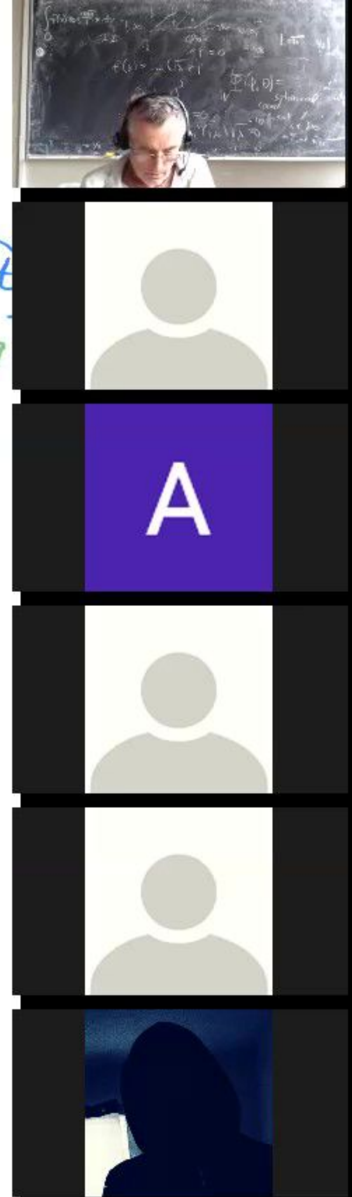
use initial conditions:

(i) $u(r,\theta,0) = 0$

(ii) $\frac{\partial u}{\partial t}(r,\theta,0) = \beta(r) \cos 5\theta$

$$0 = \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} J_m\left(\sqrt{\lambda_{mn}}\, r\right)\left(a_m \cos m\theta + b_m \sin m\theta\right) a_{mn}$$

eigenfunctions are mutually orthogonal, in particular linearly independent $\Rightarrow$ coeff. $a_{mn} = 0$ $\forall\, m,n$.



$$\beta(r)\cos 5\theta = \frac{\partial u}{\partial t}(r,\theta,0) =$$

$$= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m\left(\sqrt{\lambda_{mn}}\,r\right)\underbrace{\left(a_m\cos m\theta + b_m\sin m\theta\right)}_{b_{mn}\sqrt{\lambda_{mn}}\,c\,\cdot\,1}$$

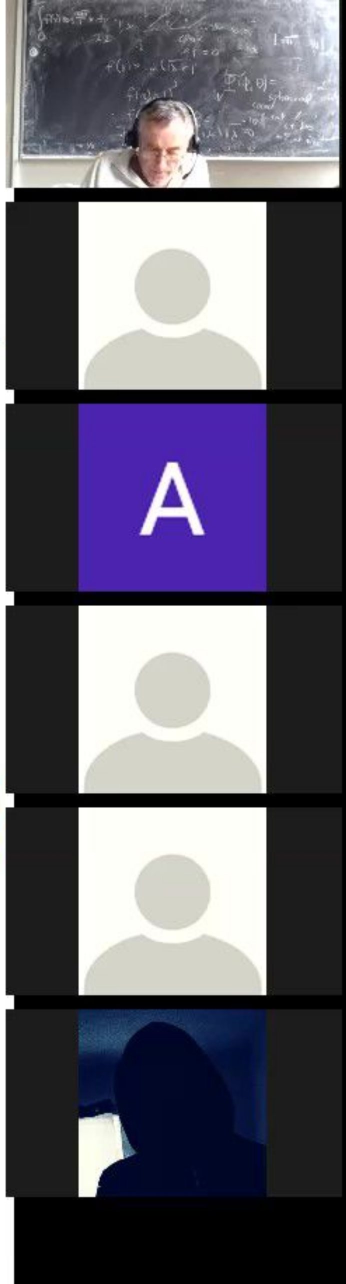
$$\frac{1}{\pi}\int_{-\pi}^{\pi}\beta(r)\cos 5\theta\cos k\theta\,d\theta = \begin{cases} 0 & \text{if } k\neq 5\\ \beta(r) & \text{if } k=5\end{cases}$$

$\boxed{\text{r.h.s}}$

multiplied r.h.s by $\cos k\theta$ and integrated

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\sum\sum J_m\Big(\underbrace{a_m\cos m\theta\cos k\theta}_{\int\,=\,0\ m\neq k} + \underbrace{b_m\sin m\theta\cos m\theta}_{\int\,=\,0}\Big)\sim$$

$$= J_k\left(\sqrt{\lambda_{k,n}}\right)\,a_k\,b_{kn}\sqrt{\lambda_{kn}}\,c$$



$$\beta(r)\cos 5\theta = \frac{\partial u}{\partial t}(r,\theta,0) =$$

$$= \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} J_m(\sqrt{\lambda_{mn}}\,r)\underbrace{(a_m\cos m\theta + b_m \sin m\theta)}_{b_{mn}\sqrt{\lambda_{mn}}\,c\,\cdot\, 1}$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\beta(r)\cos 5\theta \cos k\theta\, d\theta = \begin{cases} 0 & \text{if } k\neq 5 \\ \beta(r) & \text{if } k=5 \end{cases} =$$

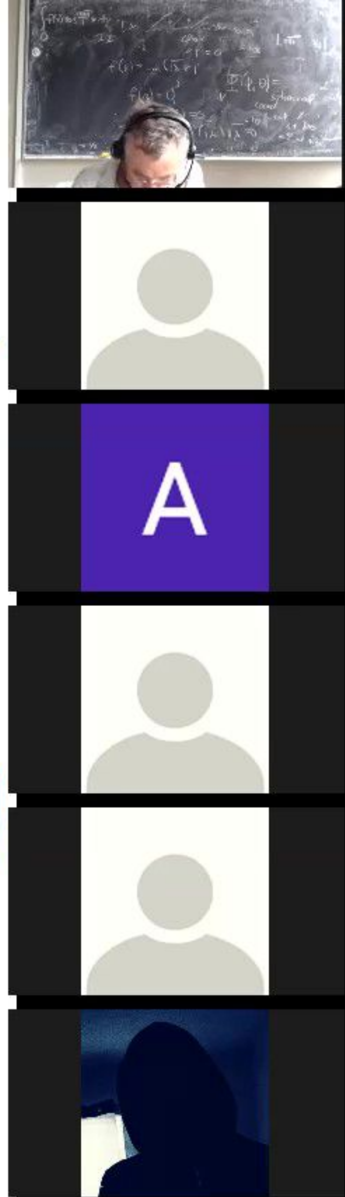
$\boxed{\text{r.h.s}}$

multiplied r.h.s by $\cos k\theta$ and integrated

$$\frac{1}{\pi}\int_{-\pi}^{\pi}\sum\sum J_m\Big(\underbrace{a_m\cos m\theta\cos k\theta}_{\int = 0\ m\neq k} + \underbrace{b_m\sin m\theta\cos m\theta}_{\int = 0}\Big)\sim$$

$$= J_k(\sqrt{\lambda_{k,n}})\,a_k\, b_{kn}\sqrt{\lambda_{kn}}\,c = 0$$

if $k\neq 5$



Result:

$$\beta(r) = \sum_{n=1}^{\infty} J_5(\sqrt{\lambda_{5n}}\, r)\, b_{5n} a_5 \sqrt{\lambda_{5n}}\, c.$$

how to calculate?

Use orthogonality relations for Bessel function

$$\Rightarrow \boxed{b_{5n} a_5 \sqrt{\lambda_{5n}}\, c = \frac{\int_0^a \beta(r) J_5(\sqrt{\lambda_{5n}}\, r)\, r\, dr}{\int_0^a J_5(\sqrt{\lambda_{5n}}\, r)^2\, r\, dr}}$$

$$\underbrace{\hphantom{xx}}_{= I(5,n)}$$

$$\Rightarrow \boxed{b_{5n} a_5 = \frac{1}{\sqrt{\lambda_{5n}}\, c} I(5,n)}$$

$$u(r,\theta,t) = \sum_{n=1}^{\infty} J_5(\sqrt{\lambda_{5n}}\, r) \cos 5\theta \, \frac{I(5,n)}{\sqrt{\lambda_{5n}}\, c} \sin \sqrt{\lambda_{5n}}\, c t$$

with

$$I(5,n) = \frac{\int_0^a \beta(r) J_5(\sqrt{\lambda_{5n}}\, r)\, r\, dr}{\int_0^a J_5(\sqrt{\lambda_{5n}}\, r)^2\, r\, dr}$$